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Massless spin-$\frac{3}{2}$ theory and the unmixed spinor representations of the Lorentz group

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Abstract. The Rarita–Schwinger theory of a massless spin-$\frac{3}{2}$ field belonging to the mixed spinor representations $(1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ and the unmixed spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ is explored. The motion is indeterminate to within a group of gauge transformations. It is found that the gauge invariants transform according to the unmixed spinor representations $(\frac{3}{2}, 0)$ and $(0, \frac{3}{2})$. The gauge invariants are quantized by a new non-canonical coordinate-covariant Lagrangian procedure, and the anticommutators are found to be positive. It is shown that the energy–momentum tensor is irremediably gauge variant, thus ruling out any possibility of gravitational interaction. It is found that electromagnetic interaction is also quite impossible to achieve.

1. Introduction

The indefinite metric problems which arise in the theory of quantized interacting spin-$\frac{3}{2}$ fields are well known (Johnson and Sudarshan 1961, Hagen 1971, 1974), as are the related causality problems of the classical theory (Velo and Zwanziger 1969, Shamaly and Capri 1972, Madore and Tait 1973, Jenkins 1974, Mainland and Sudarshan 1974, Prabhakaran *et al* 1975, Singh 1975, Tait 1975). Although it is not strictly of physical interest we judged that it would be worthwhile to study the massless spin-$\frac{3}{2}$ field equation, in order to obtain a better understanding of the nature of the singular differential operators of high-spin theory (Wightman 1973), and to obtain also a better appreciation of the roles which might be played by the various available irreducible representations (p, q) (Le Couteur 1950, Cox 1974a, b, c) of the Lorentz group.

The massless theory provides a particularly interesting test case for Fermi–Dirac quantization methods, since the field equations are indeterministic and generate constraints. We apply here the invariant quantization method of Allcock (1975a), and show that it can deal quickly and effectively with both these problems. This part of the paper can be regarded as complementary to the work of Hall (1974), which deals with the invariant quantization of constrained Bose–Einstein systems.

Throughout we use real Dirac matrices γ^μ, obeying $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, where $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Greek indices run from 0 to 3, Latin ones from 1 to 3. $\gamma^{0\mathrm{T}} = -\gamma^0$, $\gamma^{j\mathrm{T}} = \gamma^j$, $\bar{\psi}^\mu \equiv -\psi^\mu\gamma^0$ and ψ^μ is anticommutative and Hermitian.

2. The massless Rarita–Schwinger field

We take an Hermitian and anticommutative Rarita–Schwinger (Rarita and Schwinger

1941) four-vector-spinor ψ^μ to represent particles with helicity $\pm\frac{3}{2}$, and we take as Lagrangian density the Hermitian expression

$$\mathscr{L} = -\tfrac{1}{2}\mathrm{i}[\bar{\psi}_\mu(\gamma\partial)\psi^\mu - (\bar{\psi}\gamma)(\partial\psi) - \bar{\psi}^\mu\partial_\mu(\gamma\psi) - (\bar{\psi}\gamma)(\gamma\partial)(\gamma\psi)]. \tag{1}$$

We vary the action $A \equiv \int \mathscr{L}\, \mathrm{d}^4x$ to obtain the Euler–Lagrange equations

$$(\gamma\partial)\psi^\mu - \gamma^\mu(\partial\psi) - \partial^\mu(\gamma\psi) - \gamma^\mu(\gamma\partial)(\gamma\psi) = 0. \tag{2}$$

Contracting (2) with γ^μ leads us to

$$(\partial\psi) + (\gamma\partial)(\gamma\psi) = 0 \tag{3}$$

which with (2) gives

$$(\gamma\partial)\psi^\mu - \partial^\mu(\gamma\psi) = 0. \tag{4}$$

The differential operator in (2) is singular; putting $\mu = 0$ we have the primary constraint

$$\partial_j\psi_j - \gamma_j\partial_j\gamma_k\psi_k = 0. \tag{5}$$

Putting $\mu = j$ in (2) leads us to equations of motion,

$$\partial_0\psi_j - \gamma_j\partial_0\gamma_k\psi_k + \gamma_0\gamma_k\partial_k\psi_j - \gamma_0\gamma_j\partial_k\psi_k - \partial_j\psi_0 - \gamma_0\partial_j\gamma_k\psi_k + \gamma_j\gamma_k\partial_k\psi_0 + \gamma_0\gamma_j\gamma_k\partial_k\gamma_l\psi_l = 0, \tag{6}$$

which fix $\partial_0\psi_j$ completely. Differentiation of (5) with respect to x^0 and substitution from (6) for $\partial_0\psi_j$ yields no further information on the $\partial_0\psi_\mu$ and no further constraints. This means that the constraint hierarchy (Allcock 1975b) of $\mathscr{L}$ is already completed; (5) is the only constraint on the system.

2.1. *The spinor gauge group*

Superficially, it now appears that we have too many independent variables to describe massless spin-$\frac{3}{2}$ particles. However, we have equations of motion only for the ψ_j and not for ψ_0. This indeterminacy motivates us to look for a gauge group (Fierz and Pauli 1939); a group of transformations involving arbitrary functions, and expressing completely the indeterministic aspects of the dynamic evolution of the system.

Let ψ_μ undergo a spinor gauge transformation

$$\psi_\mu \to \psi_\mu + \partial_\mu\chi, \tag{7}$$

where χ is an arbitrary spinor. It is easy to show that $\mathscr{L}$ is invariant under (7), to within a perfect differential. The extremals of the action (i.e. the equations of motion) are insensitive to perfect differentials and hence the theory admits a spinor gauge group.

It is easy to see that this exhausts the indeterminacy. Moreover, from the sixteen independent variables present in ψ_μ the constraint (5) already freezes out four and the gauge freedoms of ψ_j and ψ_0 use up another eight so that only four variables are left to carry the dynamics. This is precisely the number required to describe a particle of helicity $\frac{3}{2}$ and its antiparticle of helicity $-\frac{3}{2}$.

2.2. *The canonical energy–momentum tensor*

It is well known that in the case of linear theories of particles of helicity ± 2 the appropriately linearized Einstein energy–momentum pseudotensor $t_{\mu\nu}$ is not gauge invariant, and that in spite of this a definite meaning can still be attached to the total energy–momentum carried by the field. We shall show in this subsection that a similar situation arises in the case of helicity $\pm\frac{3}{2}$, and we may plausibly infer that the same applies for all helicities h higher in absolute value than unity. Thus it appears that the theory of the electromagnetic field occupies a unique position among massless gauge theories, in that it is the only such theory for which observable concepts of energy and momentum can be defined locally. This obviously has relevance to the problem of gravitational interaction of massless particles, constituting, in effect, a no-go theorem for $|h|>1$. It also has relevance to the local properties of energy and momentum as formulated by Schwinger, for which see Bender and McCoy (1966).

We start by looking at the canonical tensor, which is defined by

$$T_{\mu\nu} \equiv \pi_{\nu\alpha}\partial_\mu\psi^\alpha - \mathscr{L}g_{\mu\nu} \tag{8}$$

and has the conservation property

$$\partial^\nu T_{\mu\nu} = 0$$

as a consequence of (2). From (1)

$$\pi_{\nu\alpha} \equiv \partial_{\mathrm{F}}\mathscr{L}/\partial(\partial^\nu\psi^\alpha) = -\tfrac{1}{2}\mathrm{i}[\bar\psi_\alpha\gamma_\nu - \bar\psi_\nu\gamma_\alpha - (\bar\psi\gamma)g_{\nu\alpha} - (\bar\psi\gamma)\gamma_\nu\gamma_\alpha], \tag{9}$$

where the suffix F denotes the right-handed derivative (cf § 3.1). Feeding (9) into (8) and using (2) we thus obtain the formula

$$T_{\mu\nu} = -\tfrac{1}{2}\mathrm{i}[\bar\psi_\alpha\gamma_\nu\partial_\mu\psi^\alpha - (\bar\psi\gamma)\partial_\mu\psi_\nu - \bar\psi_\nu\partial_\mu(\gamma\psi) - (\bar\psi\gamma)\gamma_\nu\partial_\mu(\gamma\psi)]. \tag{10}$$

From the conservation property it follows of necessity that the total energy and momentum is unaffected by the indeterminacy of the motion, i.e. is gauge invariant. Our interest is however to see whether $T_{\mu\nu}$ itself is gauge invariant.

Under the gauge transformation (7), $T_{\mu\nu}$ changes in a non-trivial way. We find that it transforms according to

$$T_{\mu\nu} \to T_{\mu\nu} + \partial^\lambda(\mathscr{T}_{\mu\nu\lambda} - \mathscr{T}_{\mu\lambda\nu}) \tag{11a}$$

where

$$\mathscr{T}_{\mu\nu\lambda} = -\tfrac{1}{2}\mathrm{i}\{\tfrac{1}{2}\bar\chi\gamma_\nu\gamma_\lambda\partial_\mu[(\gamma\partial)\chi + (\gamma\psi)] + \bar\chi\gamma_\nu\partial_\mu(\partial_\lambda\chi + \psi_\lambda) + \partial_\mu\bar\chi\gamma_\lambda[\psi_\nu + \tfrac{1}{2}\gamma_\nu(\gamma\psi)]\}. \tag{11b}$$

This means that the energy and momentum cannot be localized in space in any way independent of the choice of gauge, although the explicit antisymmetry built into (11a) makes it obvious that the *total* energy and momentum is indeed gauge invariant, in accord with the general argument given above.

2.3. *Belinfante's tensor*

We now look at Belinfante's symmetric energy–momentum tensor to see if this fares any better.

Under the infinitesimal Lorentz transformation $x_\mu \to x'_\mu = x_\mu + \delta\omega_{\mu\nu}x^\nu$ we have

$$\psi_\mu(x) \to \psi'_\mu(x') = \psi_\mu(x) + \tfrac{1}{2}\delta\omega_{\rho\sigma}M^{\rho\sigma}{}_\mu{}^\nu\psi_\nu(x), \tag{12}$$

where

$$M^{\rho\sigma}{}_\mu{}^\nu = -\tfrac{1}{4}[\gamma^\rho, \gamma^\sigma]\delta_\mu{}^\nu + \delta_\mu{}^\rho g^{\sigma\nu} - \delta_\mu{}^\sigma g^{\rho\nu} \tag{13}$$

(Le Couteur 1950). The spin moment of momentum density $H^{\nu\rho\mu}$ is defined by

$$H^{\nu\rho\mu} \equiv \pi^{\mu\alpha} M^{\nu\rho}{}_{\alpha}{}^{\beta}\psi_\beta \tag{14}$$

where $\pi^{\mu\alpha}$ is defined in (9).

Belinfante's symmetrized energy–momentum tensor $\theta^{\nu\rho}$ is then defined by

$$\theta^{\nu\rho} \equiv T^{\nu\rho} - \tfrac{1}{2}\partial_\mu (H^{\nu\rho\mu} - H^{\nu\mu\rho} - H^{\rho\mu\nu}) \tag{15}$$

(Belinfante 1939, J M Jauch in Wentzel 1949). From (9), (13) and (14) we have

$$\begin{aligned} H^{\nu\rho\mu} = -\tfrac{1}{2}\mathrm{i}\{ & -\tfrac{1}{4}\bar{\psi}^\alpha\gamma^\mu[\gamma^\nu, \gamma^\rho]\psi_\alpha + 2\bar{\psi}^\nu\gamma^\mu\psi^\rho + \tfrac{1}{2}\bar{\psi}^\mu[\gamma^\nu, \gamma^\rho](\gamma\psi) \\ & -(\bar{\psi}\gamma)g^{\mu\nu}\psi^\rho + (\bar{\psi}\gamma)g^{\mu\rho}\psi^\nu + \tfrac{1}{4}(\bar{\psi}\gamma)\gamma^\mu[\gamma^\nu, \gamma^\rho](\gamma\psi)\}. \end{aligned} \tag{16}$$

From (10), (15) and (16) we eventually arrive at

$$\begin{aligned} \theta^{\nu\rho} = -\tfrac{1}{2}\mathrm{i}[& \tfrac{1}{2}\bar{\psi}^\alpha(\gamma^\nu\partial^\rho + \gamma^\rho\partial^\nu)\psi_\alpha + \tfrac{1}{2}(\bar{\psi}\gamma)(\partial^\nu\psi^\rho + \partial^\rho\psi^\nu) - \tfrac{1}{2}(\bar{\psi}^\nu\partial^\rho + \bar{\psi}^\rho\partial^\nu)(\gamma\psi) \\ & + \tfrac{1}{2}(\bar{\psi}^\rho\gamma^\nu + \bar{\psi}^\nu\gamma^\rho)(\gamma\partial)(\gamma\psi) - \bar{\psi}^\mu\partial_\mu(\gamma^\nu\psi^\rho + \gamma^\rho\psi^\nu) + (\bar{\psi}^\nu\partial^\rho + \bar{\psi}^\rho\partial^\nu)(\partial\psi) \\ & - \bar{\psi}^\alpha(\gamma\partial)\psi_\alpha g^{\nu\rho} - (\bar{\psi}\gamma)(\partial\psi)g^{\nu\rho} + 2\bar{\psi}^\mu\partial_\mu(\gamma\psi)g^{\nu\rho}] \end{aligned} \tag{17}$$

after having made much use of the equations of motion (2). The symmetry of $\theta^{\nu\rho}$ can be plainly seen and a little calculation reveals that $\partial_\rho\theta^{\nu\rho} = 0$ by virtue of (2).

Let ψ^μ be once more subjected to the gauge transformation (7). The extra terms produced in $\theta^{\nu\rho}$ are again non-trivial. After much calculation involving the repeated use of (2) they can be cast into the form

$$\partial_\alpha\partial_\beta\mathcal{S}^{\nu\rho\alpha\beta} \equiv \tfrac{1}{4}\mathrm{i}\partial_\alpha\partial_\beta(g^{\rho\beta}S^{\nu\alpha} + g^{\nu\beta}S^{\rho\alpha} - g^{\nu\rho}S^{\alpha\beta} - g^{\alpha\beta}S^{\nu\rho}) \tag{18}$$

where

$$S^{\rho\nu} \equiv -\bar{\chi}[\gamma^\rho\psi^\nu + \gamma^\nu\psi^\rho - g^{\nu\rho}(\gamma\psi) + \tfrac{1}{2}(\gamma^\nu\partial^\rho + \gamma^\rho\partial^\nu)\chi - \tfrac{1}{2}g^{\nu\rho}(\gamma\partial)\chi] = S^{\nu\rho}.$$

The structure $\mathcal{S}^{\nu\rho\alpha\beta}$ can be seen to vanish when any three of its indices are put equal; in particular if $\alpha = \beta = \nu = 0$. Therefore, in view of the symmetry of $\mathcal{S}$ in the indices $\alpha\beta$, we have

$$\int \partial_\alpha\partial_\beta\mathcal{S}^{0\rho\alpha\beta}\,\mathrm{d}^3x = \int \partial_j(\partial_\beta\mathcal{S}^{0\rho j\beta} + \partial_0\mathcal{S}^{0\rho j 0})\,\mathrm{d}^3x = 0.$$

Thus the total amount of energy and momentum carried by $\theta^{\nu\rho}$ is gauge invariant, as it should be (and it is of course equal to that carried by $T^{\nu\rho}$). But the distribution of this energy and momentum through space depends in an essential way upon the choice of gauge, as was the case with $T^{\nu\rho}$. There is no covariant way to fix the gauge and thus dispose of this unwelcome and unphysical ambiguity.

2.4. *Perfect differentials*

The lack of gauge invariance of the energy–momentum tensors and the non-sensitivity of the action principle to perfect differentials prompts us to ask whether gauge invariance could be restored by means of adding perfect differentials to $\mathcal{L}$. There are three perfect differentials we could consider, namely

$$\partial_\nu(\bar{\psi}^\mu\gamma^\nu\psi_\mu), \qquad \partial_\nu((\bar{\psi}\gamma)\gamma^\nu(\gamma\psi)), \qquad \partial_\mu(\bar{\psi}^\mu(\gamma\psi)).$$

The first two of these are identically zero by virtue of the anticommutative property of the ψ, which leaves only the third. Let us therefore consider

$$\tfrac{1}{2}\mathrm{i}k\partial_\mu(\bar{\psi}^\mu(\gamma\psi)) \equiv \tfrac{1}{2}\mathrm{i}k\bar{\psi}^\mu\partial_\mu(\gamma\psi) - \tfrac{1}{2}\mathrm{i}k(\bar{\psi}\gamma)(\partial\psi), \tag{19}$$

where k is an arbitrary constant.

Its contribution to $\pi^{\mu\alpha}$ is $\pi'^{\mu\alpha} = \frac{1}{2}\mathrm{i}k\bar{\psi}^\mu\gamma^\alpha - \frac{1}{2}\mathrm{i}k(\bar{\psi}\gamma)g^{\mu\alpha}$ and hence its contribution to $T_{\mu\nu}$ is

$$T'_{\mu\nu} = \tfrac{1}{2}\mathrm{i}k\bar{\psi}_\nu\partial_\mu(\gamma\psi) - \tfrac{1}{2}\mathrm{i}k(\bar{\psi}\gamma)\partial_\mu\psi_\nu - \tfrac{1}{2}\mathrm{i}k\bar{\psi}^\lambda\partial_\lambda(\gamma\psi)g_{\mu\nu} + \tfrac{1}{2}\mathrm{i}k(\bar{\psi}\gamma)(\partial\psi)g_{\mu\nu} \tag{20}$$

from (8). The structure of this expression is quite different from that of (11*a*), (11*b*); therefore there is no value of k which would render $T_{\mu\nu}$ gauge invariant.

From (13) and (14) the contribution of (19) to $H^{\nu\rho\mu}$ is

$$H'^{\nu\rho\mu} = \tfrac{1}{2}\mathrm{i}k[(\bar{\psi}\gamma)(g^{\mu\rho}\psi^\nu - g^{\mu\nu}\psi^\rho)],$$

and so

$$-\tfrac{1}{2}\partial_\mu(H'^{\nu\rho\mu} - H'^{\nu\mu\rho} - H'^{\rho\mu\nu}) = -\tfrac{1}{2}\mathrm{i}k\partial_\mu[(\bar{\psi}\gamma)(g^{\rho\nu}\psi^\mu - g^{\mu\nu}\psi^\rho)].$$

Adding this to (20) we have that the contribution to $\theta^{\nu\rho}$ is zero, so obviously $\theta^{\nu\rho}$ cannot be made gauge invariant either.

2.5. *The chirality current*

The Lagrangian (1) is invariant under the chiral transformation $\psi_\mu \to \exp(\theta\gamma_5)\psi_\mu$ and hence admits, through Noether's theorem, a conserved chiral current

$$J_\mu = \tfrac{1}{2}\mathrm{i}\bar{\psi}_\alpha\gamma_\mu\gamma_5\psi^\alpha - \mathrm{i}(\bar{\psi}\gamma)\gamma_5\psi_\mu + \tfrac{1}{2}\mathrm{i}(\bar{\psi}\gamma)\gamma_\mu\gamma_5(\gamma\psi).$$

It can be shown that J_μ is gauge variant and insensitive to the perfect differential (19), and that the associated chiral charge is gauge invariant. Thus J_μ and $\theta_{\mu\nu}$ have very similar gauge properties.

2.6. *Gauge invariants*

We are now prompted to ask what are the gauge invariants of this theory.

Note that the left-hand sides of equations (2), (3), (4) and (5) are all gauge invariants, but are nullified by the dynamics. A set of non-zero gauge invariants is the set

$$F_{\mu\nu} \equiv \partial_\nu\psi_\mu - \partial_\mu\psi_\nu. \tag{21}$$

By Poincaré's lemma, the set $F_{\mu\nu}(x)$ determines $\psi_\mu(x)$ to within a gauge transformation (7) and is therefore complete. In terms of $F_{\mu\nu}$ the field equations (2) read

$$\gamma^\nu F_{\mu\nu} + \tfrac{1}{2}\gamma_\mu\gamma^\lambda\gamma^\nu F_{\lambda\nu} = 0 \tag{22}$$

whence

$$\gamma^\nu F_{\mu\nu} = 0. \tag{23}$$

From (23) it is clear that the spatial components F_{jk} are already complete, modulo the field equations.

2.7. Causality

We can examine the causality properties in several ways. One way is to exploit the freedom of gauge and choose one in which the field equations (4) take a simple form.

Consider a gauge transformation of $(\gamma\psi)$: $(\gamma\psi)\rightarrow(\gamma\psi)+(\gamma\partial)\chi$. We can choose χ to make $(\gamma\psi)$ zero because $(\gamma\partial)$ is a deterministic wave operator. In this gauge (4) reduces to $(\gamma\partial)\psi_\mu=0$ which is manifestly deterministic.

Alternatively, consider the equations of motion of the $F_{\mu\nu}$, which from (21) and (23) are

$$\gamma^\nu F_{\mu\nu}=0, \qquad F_{\mu\nu}=-F_{\nu\mu}, \tag{24}$$

$$\partial_\rho F_{\mu\nu}+\partial_\nu F_{\rho\mu}+\partial_\mu F_{\nu\rho}=0. \tag{25}$$

These are readily shown to constitute a completely deterministic system. Thus, although the *whole* theory is partly acausal we have now accounted for all the acausal features, in that we have found both the complete gauge group and a complete set of gauge invariants.

3. Invariant quantization

We shall quantize the theory directly, without invoking canonical concepts, by using the coordinate-invariant method described by Allcock (1975a) and Hall (1974), to whom the reader is referred for further details and consistency proofs.

To the extent that one works with a general gauge it is neither meaningful nor possible to quantize gauge variant quantities—only the gauge invariants can be quantized and so we work with the complete set (21) and not with the potentials ψ_μ.

3.1. General theory

Let G_1 and G_2 be even gauge invariants of the theory. Consider the infinitesimal displacements of the solution, $\delta_1\psi_\mu$, generated by ϵG_1, where ϵ is an infinitesimal.

The main generating equation for $\delta_1\psi_\nu$ is

$$\epsilon\int\frac{\mathscr{d}_{\mathrm F}G_1}{\mathscr{d}\psi_\nu(x')}\delta\psi_\nu(x')\,\mathrm d^3x'+\iint\lambda_a(x')\frac{\mathscr{d}_{\mathrm F}\xi_a(x')}{\mathscr{d}\psi_\nu(x'')}\delta\psi_\nu(x'')\,\mathrm d^3x'\,\mathrm d^3x''$$

$$=-\iint\left(\frac{\mathscr{d}_{\mathrm F}\pi^0{}_\alpha(x'')}{\mathscr{d}\psi^\beta(x')}+\frac{\mathscr{d}_{\mathrm F}\pi^0{}_\beta(x')}{\mathscr{d}\psi^\alpha(x'')}\right)\delta_1\psi^\beta(x')\delta\psi^\alpha(x'')\,\mathrm d^3x'\,\mathrm d^3x'', \tag{26}$$

where $\xi_a(x)$ are the constraint functions (5) and $\lambda_a(x)$ are continuous (anticlassical, imaginary) infinitesimal Lagrange multipliers so that the auxiliary variations $\delta\psi_\nu$ can be regarded as being free, and where $\mathscr{d}_{\mathrm F}G/\mathscr{d}\psi_\nu(x')$ is the *right-handed* anticlassical or Fermi–Dirac functional derivative defined by the factor ordering

$$\delta G=\int\frac{\mathscr{d}_{\mathrm F}G[\psi]}{\mathscr{d}\psi_\nu(x')}\delta\psi_\nu(x')\,\mathrm d^3x'. \tag{27}$$

To the main equation (26) we must adjoin the equations of constraint which imply that

$$\int\frac{\mathscr{d}_{\mathrm F}\xi_a(x)}{\mathscr{d}\psi_\nu(x')}\delta_1\psi_\nu(x')\,\mathrm d^3x'=0, \qquad \forall\boldsymbol{x}. \tag{28}$$

In all these equations spinor indices may properly be taken into account by regarding α, β, etc as compound indices. Now there is a general theorem (Allcock 1975a, § 3.7, 1975b, theorem 2) which guarantees that the joint system (26), (28) will have a solution $\delta_1\psi_\nu$ for all allowed initial data (i.e. data obeying the constraints $\xi_a = 0$). Moreover, in the same works it is proved that every freedom of the solution (i.e. every solution obtained by letting $G_1 = 0$) will represent an infinitesimal gauge freedom, and that the gauge freedoms so found will exhaust the infinitesimal gauge group.

The equal-time Poisson bracket of any two even gauge invariants G_1 and G_2 may therefore be unambiguously defined by either of the equivalent formulae

$$(G_1, G_2) = -\epsilon^{-1} \int \frac{\mathscr{d}_F G_2}{\mathscr{d}\psi_\nu(x'')} \delta_1\psi_\nu(x'')\, d^3x'' \equiv -\epsilon^{-1}\delta_1 G_2$$

$$= +\epsilon^{-1} \int \frac{\mathscr{d}_F G_1}{\mathscr{d}\psi_\nu(x'')} \delta_2\psi_\nu(x'')\, d^3x'' \equiv +\epsilon^{-1}\delta_2 G_1 \tag{29}$$

where $\delta_2\psi_\nu$ is the solution of (26) and (28) with G_1 replaced by G_2. The Poisson bracket introduced in this way may be carried over to quantum theory by Dirac's rule of correspondence

$$i(G_1, G_2) = G_1G_2 - G_2G_1.$$

However, the system under consideration is of Fermi–Dirac type, requiring anticommutators. We therefore use Peierls' device, putting

$$G_1 = i\Upsilon_1 F_1, \qquad G_2 = i\Upsilon_2 F_2$$

where Υ_1, Υ_2 are anticommuting constants (Allcock 1975a) and F_1 and F_2 are odd gauge invariants. We then define the anticlassical analogue of the quantum anticommutator by writing

$$-i\Upsilon_1\Upsilon_2\{F_1, F_2\} = (i\Upsilon_1F_1, i\Upsilon_2F_2). \tag{30}$$

The Poisson bracket on the right-hand side in (30) can now be calculated using the above procedure, after which the arbitrary anticommuting constants can be taken to the left and cancelled. The rule of correspondence for the symmetric structure remaining is

$$\{F_1, F_2\} = F_1F_2 + F_2F_1. \tag{31}$$

3.2. *Application to the case in hand*

With these necessary preliminaries now set out we return to the massless spin-$\frac{3}{2}$ field, writing spinor indices explicitly for clarity from now on. For G_1 we take

$$G_1 = i\Upsilon_1F_1 = i\Upsilon_1(\partial_k\psi_j - \partial_j\psi_k)_c$$

for which, by comparison with (27), we have

$$\frac{\mathscr{d}_F G_1}{\mathscr{d}\psi_{la}(x')} = i\Upsilon_1(\delta_{kl}\partial_j' - \delta_{jl}\partial_k')\delta^3(x - x')\delta_{ca}. \tag{32}$$

The system only has one continuum of constraint equations (5), namely

$$\xi_b(x) \equiv (\partial_j\psi_j - \gamma_j\partial_j\gamma_k\psi_k)_b = 0,$$

whence

$$\frac{d_{\rm F}\xi_b(x')}{d\psi_{la}(x'')}=[-\partial_l''\delta_{ba}+(\gamma_m\partial_m''\gamma_l)_{ba}]\delta^3(x'-x''). \tag{33}$$

From (9)

$$\frac{d_{\rm F}\pi^0{}_{\beta b}(x')}{d\psi^\alpha{}_a(x'')}=-\tfrac{1}{2}\mathrm{i}(g_{\alpha\beta}+g_{0\alpha}\gamma_0\gamma_\beta+g_{0\beta}\gamma_0\gamma_\alpha+\gamma_0\gamma_\alpha\gamma_0\gamma_\beta)_{ab}\delta^3(x'-x'')$$

$$=-\tfrac{1}{4}\mathrm{i}[\gamma_\alpha,\gamma_\beta]_{ab}(1-\delta_{\alpha 0})(1-\delta_{\beta 0})\delta^3(x'-x'').$$

Therefore we can write the right-hand side of (26) as

$$\tfrac{1}{2}\mathrm{i}\int[\gamma_l,\gamma_m]_{ab}\delta_1\psi'_{mb}\delta\psi'_{la}\,\mathrm{d}^3x'. \tag{34}$$

Substituting into (26) for all the above quantities we have, on integrating over δ functions and cancelling the auxiliary variation $\delta\psi_l'$,

$$\tfrac{1}{2}\mathrm{i}[\gamma_l,\gamma_m]_{ab}\delta_1\psi'_{mb}=\mathrm{i}\epsilon\Upsilon_1(\delta_{kl}\partial_j'-\delta_{jl}\partial_k')\delta^3(x-x')\delta_{ac}-\delta_{ba}\partial_l'\lambda_b'+(\gamma_m\gamma_l)_{ba}\partial_m'\lambda_b',$$

or

$$\tfrac{1}{2}\mathrm{i}[\gamma_l,\gamma_m]_{ab}(\delta_1\psi'_{mb}+\mathrm{i}\partial_m'\lambda_b')=\mathrm{i}\epsilon\Upsilon_1(\delta_{kl}\partial_j'-\delta_{jl}\partial_k')\delta^3(x-x')\delta_{ac}. \tag{35}$$

It is easy to show that

$$\tfrac{1}{2}\gamma_l\gamma_r\,.\,\tfrac{1}{2}[\gamma_m,\gamma_l]=\delta_{rm},$$

therefore (35) can be solved:

$$\delta_1\psi'_{ra}=-\tfrac{1}{2}\epsilon\Upsilon_1(\gamma_k\gamma_r\partial_j'-\gamma_j\gamma_r\partial_k')_{ac}\delta^3(x-x')-\mathrm{i}\partial_r'\lambda_a'. \tag{36}$$

The constraint equations (28) read as

$$[\gamma_l,\gamma_m]_{ab}\partial_l'\delta_1\psi'_{mb}=0 \tag{37}$$

and are satisfied no matter what the value of λ. Thus λ is free and this freedom in the solution $\delta_1\psi_r$ corresponds exactly to the fact that ψ_r is gauge variant. Evidently λ represents an infinitesimal displacement of gauge, a solution of (26) with G_1 set equal to zero. Moreover, $\delta\psi_0$ does not appear at all in the generating equations so that a second gauge freedom arises on this count. These two gauge freedoms are the only ones arising in the generating equations. Thus, at fixed time, we have just two independent continuous spinor-valued freedoms of solution, namely $\delta_1\psi_r\to\delta_1\psi_r-\mathrm{i}\partial_r\lambda$ and $\delta_1\psi_0\to\delta_1\psi_0+\delta S$, where δS is arbitrary. For variable time the field equations (2) give us that $\delta S=-\mathrm{i}\partial_0\lambda$.

Thus the gauge freedoms found in the generating equations are indeed entirely equivalent to those found directly from the field equations, in accord with the general theory of Lagrangian systems.

We can now use (36) in (29). Taking

$$G_2=\mathrm{i}\Upsilon_2F_2=\mathrm{i}\Upsilon_2(\partial_q'\psi_p'-\partial_p'\psi_q')_d,$$

we evaluate δ_1G_2 and hence $(G_1,G_2)\equiv-\mathrm{i}\Upsilon_1\Upsilon_2\{F_1,F_2\}$. The λ term disappears since curl grad $\lambda=0$. We obtain finally

$$\{(\partial_k\psi_j-\partial_j\psi_k)_c,(\partial_q'\psi_p'-\partial_p'\psi_q')_d\}$$
$$=-\tfrac{1}{2}[(\gamma_k\gamma_q)_{dc}\partial_j\partial_p-(k\leftrightarrow j)-(p\leftrightarrow q)+(k\leftrightarrow j,p\leftrightarrow q)]\delta^3(x-x'). \tag{38}$$

This is symmetric as required since $(\gamma_k\gamma_q)_{dc} = (\gamma_q\gamma_k)_{cd}$. Any other equal-time anticommutators can be obtained from (38) by exploiting (23).

3.3. *Positivity*

It is sufficient to consider Fourier components with the momentum along the 1-axis. It is then apparent that the anticommutators are characterized by the eight-dimensional matrix

$$\frac{1}{2}\begin{pmatrix} \gamma_2\gamma_2 & \gamma_2\gamma_3 \\ \gamma_3\gamma_2 & \gamma_3\gamma_3 \end{pmatrix}.$$

Since this matrix is real, symmetric and idempotent it is also positive. It is not positive definite because of the constraint $\gamma_j\gamma_k(\partial_j\psi_k - \partial_k\psi_j) = 0$, which follows from (24) or (5).

5. Minimal interaction

Incipient Čerenkov radiation makes it impossible for massless charged particles to exist in the physical world. It is nevertheless instructive to consider motion in an external electromagnetic field, and to see what happens in the present case if we try to introduce minimal electromagnetic interaction. The greatest problem arises when we come to consider the spinor gauge transformation (7). Obviously we require invariance under the *electromagnetic* gauge transformation, and this leads us to write the *spinor* gauge transformation as $\psi_\mu \to \psi_\mu + \mathrm{D}_\mu\chi$, where D_μ is the usual electromagnetic derivative. We now find that $\mathscr{L}$ is not invariant under this transformation, even to within a perfect differential, because of the non-commutativity of the D_μ, and that no additional terms can be added to $\mathscr{L}$ to restore gauge invariance. A Lagrangian theory of massless spin-$\frac{3}{2}$ particles with minimal electromagnetic interaction is therefore not possible.

5. Conclusions

From (24) we can easily see that $F_{\mu\nu}$ belongs to an irreducible representation of the Lorentz group. It obeys $\gamma^\nu F_{\mu\nu} = 0$, $F_{\mu\nu} = -F_{\nu\mu}$ giving eight free components. It is therefore quite clear that it belongs to the real representation $(\frac{3}{2}, 0)\oplus(0, \frac{3}{2})$ (Le Couteur 1950).

The various representations (p, q) of the Lorentz group can be plotted on a diagram as in figure 1 (Le Couteur 1950). Only those involved in Fermi–Dirac systems are shown. Paired representations straddling the diagonal attain positivity. The Rarita–Schwinger vector-spinor belongs to $(1, \frac{1}{2})\oplus(\frac{1}{2}, 1)\oplus(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$ which is a direct sum of two such paired representations. In the present work we have found a use for the representations at the edge of the diagram, namely $(p, 0)$ and $(0, q)$; these are the gauge invariants of the massless theory. This also intimates the use of the representations (p, q) $(p+q = s, |p-q| \neq \frac{1}{2}, p \neq 0, q \neq 0)$, such as $(2, \frac{1}{2})$ in spin-$\frac{5}{2}$ theory. These are intermediaries in the chain of differentiations leading from the variables of the action principle to the gauge invariants. Thus a definite descriptive role for these representations appears for the first time in massless theories, although it is obvious that they cannot be incorporated into the massless action principle itself except by the use of dimensional constants, which we deem inappropriate. We take our work as confirming

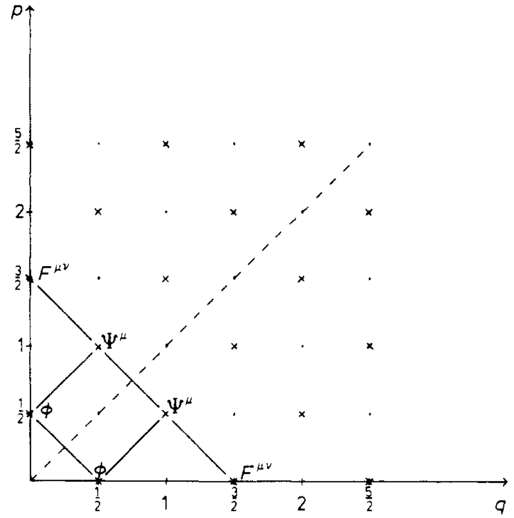

Figure 1. The representations of the Lorentz group involved in Fermi–Dirac systems. $F^{\mu\nu} \equiv \partial^\nu\psi^\mu - \partial^\mu\psi^\nu$, $\gamma_\mu F^{\mu\nu} = 0$, $\Psi^\mu \equiv \psi^\mu + \frac{1}{4}\gamma^\mu(\gamma\psi)$, $\phi \equiv (\gamma\psi)$.

the essential correctness of the Rarita–Schwinger or Fierz–Pauli approach for massless particles, in which the representations away from the diagonal play no part in the Lagrange function itself.

Previous workers (Johnson and Sudarshan 1961, Hurley 1972, Fisk and Tait 1973) as well as ourselves have tried, as yet to no avail, to incorporate the representation $(\frac{3}{2}, 0)\oplus(0, \frac{3}{2})$ into the massive spin-$\frac{3}{2}$ Lagrangian in a satisfactory manner. We have felt for some time that this representation may perhaps have some dynamic role to play in the massive case, but have not yet reached an understanding of what this role might be. But we find considerable encouragement for further effort in this direction in the results reported above. At the same time we remark that our work indicates the very privileged position enjoyed by massless particles of spins 0, $\frac{1}{2}$ and 1, these being the only massless particles for which energy and momentum can be defined locally.

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